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Generalization of Vinen's equation including transition to superfluid turbulence

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Abstract
A phenomenological generalization of the well known Vinen equation for the evolution of vortex line density in superfluid counterflow turbulence is proposed. This generalization includes nonlinear production terms in the counterflow velocity and corrections depending on the diameter of the tube. The equation provides a unified framework for the various phenomena (stationary states and transitions) present in counterflow superfluid turbulence: in fact, it is able to describe the laminar regime, the first-order transition from laminar to turbulent TI state, the two turbulent states, the transition from TI to TII turbulent states, and it yields a slower decay of the counterflow turbulence than the classical local description. Finally, a comparison with the experimental results shows that the contribution of the new terms is prevalent in the laminar and in the turbulent TI regime, while in the fully developed turbulent TII regime the equation reduces to the original Vinen equation.

1. Introduction

The description of superfluid turbulence is a stimulating challenge for statistical physics and hydrodynamics. Superfluid turbulence can be understood in terms of a random tangle of quantized vortex lines in (the superfluid component of) liquid helium II [1–4]. In experiments where the turbulence is generated by thermal counterflow in a tube of circular cross section, the vortex line density in such a tangle is observed to develop from a low-density state (TI) to a higher-density state (TII) that can be associated with the homogeneous state [5, 6]. In some occasions, one observes that the transition from laminar to turbulent regimes admits metastable laminar regions [5, 6].

In his pioneering work in this field, following a suggestion of Feynman [7], Vinen [8] identified the dissipation associated with the turbulence with the mutual friction between the vortex lines and the normal-fluid component and derived his famous evolution equation for

L, the average vortex line length for unit volume (briefly called *vortex line density*), using the known results concerning the dynamics of a single vortex line and the analogy with classical turbulence. This equation describes the second turbulent regime satisfactorily, but not the laminar one and the two transitions (laminar–TI and TI–TII). In a previous work [9] a first generalization of Vinen's equation for the evolution of vortex tangle in superfluid turbulence has been proposed, taking into account the influence of a non-vanishing ratio between the average separation between vortex lines $\delta \simeq L^{-1/2}$ and the diameter d of the channel. This equation allows us to describe the transition from TI to TII turbulent regimes, but not the laminar–TI transition.

Our aim here is to write an equation for the evolution of vortex tangle in superfluid turbulence, able to describe the three regimes observed in counterflow, in tubes of diameter d, namely, a laminar regime at low counterflow velocity V, followed by two turbulent regimes TI and TII at increasing values of V. By simple dimensional arguments and outlining a possible physical basis, an evolution equation for the vortex line density L is proposed, which includes both Vinen's correction to the vortex generation term depending on the diameter d of the tube and Vinen's alternative contribution (which is quadratic in the counterflow velocity V); further, in order to describe the transition from TI to TII states at the second critical velocity, a steep change in the original Vinen production term is introduced. The generalized equation is able to describe in a simple way the three stationary regimes observed in counterflow experiments, in contrast to Vinen's equation, which is restricted to fully developed counterflow turbulence (turbulent TII regime) and it yields a slower decay of the counterflow turbulence than the classical local description.

Our description of these phenomena (stationary states and transitions) is purely phenomenological, but it provides, for the first time, at our knowledge, a unified framework for the various phenomena present in counterflow superfluid turbulence.

We will not consider, in the paper, the influence of rotation on counterflow superfluid turbulence and eventual phenomena of intermittence, studied in [10] and in [11].

The plan of the paper is as follows. In section 2 a brief review of counterflow superfluid turbulence is given; in section 3 the generalized equation for the evolution of vortex line density is written and a microscopic qualitative description of the transition phenomena is proposed. Sections 4 and 5 are devoted to the study of the stationary solutions of the new equation and to a comparison with experimental results. In section 6, finally, the decay of vortices in the absence of the counterflow is studied.

2. Brief description of superfluid turbulence

A typical way to produce superfluid turbulence consists of a channel heated at a closed end, subject to a heat flux $\mathbf{q}$, exceeding a critical value $\mathbf{q}_c$. This experimental situation, characterized by no net matter flow but only heat transport, is called thermal counterflow. In these experiments a random array of vortex filaments is present, which produces a damping force, known as mutual friction force. The measurements of vortex lines are described as giving a macroscopic average of the vortex line length per unit volume L. There are essentially two methods to measure the line density L in superfluid ^{4}He: observations of temperature gradients in the channel and observations of changes in the attenuation of the second sound waves [1–4].

The best known model of superfluid helium is the two-fluid model [12, 13], which regards helium II as a mixture of normal fluid and superfluid, with densities ρ_n and ρ_s respectively and velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ respectively, with total density ρ and velocity $\mathbf{v}$ defined by $\rho = \rho_s + \rho_n$ and $\rho\mathbf{v} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n$. The normal component behaves as a normal fluid, with normal viscosity and thermal conductivity; the superfluid component is an ideal fluid, which neither experiences dissipation nor carries entropy. In this framework, the superfluid turbulence is described in terms of a random tangle of quantized vortex lines in the superfluid component of liquid helium II.

An alternative model of superfluid helium is the one-fluid model [14] based on extended thermodynamics [15]. In this model, the heat flux $\mathbf{q}$ takes the place of the relative velocity $\mathbf{V} = \mathbf{v}_n - \mathbf{v}_s$. In fact, in the two-fluid model, these two quantities are linked by the relation $\mathbf{q} = \rho_s T s \mathbf{V}$, with s the entropy per unit mass and T the temperature. In this new framework the vortex array is described by introducing a vorticity tensor $\mathbf{P}_\omega$ [11], whose trace is proportional to the total length of vortices per unit of volume, L.

As observed by Donnelly in [1, 2], counterflow experiments have a history of many decades, but much remains to be explained in these experiments. The best known equation in the field of superfluid turbulence is Vinen's equation [8], which describes the evolution of L, the total length of vortex lines per unit volume, in counterflow situations characterized by a relative velocity V of normal fluid with respect to superfluid component. Vinen suggested that in homogeneous counterflow turbulence there is a balance between generation and decay processes, which leads to a steady state of quantum turbulence in the form of a self-maintained vortex tangle. The form of Vinen's equation is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \alpha V L^{3/2} - \beta\kappa L^2, \tag{2.1}$$

with α and β dimensionless constants and κ the quantum of vorticity, ascribed by $\kappa = h/m$, with m the mass of the ^{4}He atoms and h Planck's constant ($\kappa = 9.97 \times 10^{-4}$ cm^2 s^{-1}).

Vinen's original derivation of equation (2.1) relies on dimensional analysis and on physical, observational and statistical ingredients. He considered homogeneous superfluid turbulence and assumed that the time derivative $\mathrm{d}L/\mathrm{d}t$ is composed of two terms:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_f - \left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_d, \tag{2.2}$$

the first responsible for the growth of L, the second for its decay. Vinen assumes that the term $[\mathrm{d}L/\mathrm{d}t]_f$ depends on the quantum of circulation κ, the instantaneous value of L and the force $\mathbf{f}$ between the vortex line and the normal component, which is linked to the intensity V of the counterflow velocity $\mathbf{V}$; dimensional analysis leads to the equation [8, 16, 17]

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_f = \kappa L^2 \phi_f \left[\frac{V}{\kappa L^{1/2}}\right], \tag{2.3}$$

where ϕ_f is some dimensionless function of its dimensionless argument. The determination of this function is one of the most delicate problems of the phenomenological theory [16, 17]. Vinen, by analogy with the growth of a vortex ring, assumed that the dimensionless function ϕ_f depends linearly on its argument, obtaining

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_f = \alpha V L^{3/2}, \tag{2.4}$$

with α a dimensionless constant. Another possibility is to assume that ϕ_f is quadratic on its arguments [17], which leads to

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_f = \alpha' \frac{V^2}{\kappa} L. \tag{2.5}$$

This form of the term responsible for the growth of vortices is known as Vinen's alternative production term. This latter possibility is discussed at length in [17].

The form of the $[\mathrm{d}L/\mathrm{d}t]_d$ term, responsible for the vortex decay, was determined by Vinen in analogy with classical turbulence. He assumed that Feynman's model of vortex breakup is analogous to Kolmogorov's cascade in classical turbulence, obtaining

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_d = -\beta\kappa L^2 \qquad \text{with } \beta = \frac{\chi_2}{2\pi}, \tag{2.6}$$

χ_2 being a dimensionless constant of the order of unity [8, 17].

From (2.4) and (2.6) Vinen's equation (2.1) follows immediately, while, using equation (2.5) for the production term, one obtains the alternative equation:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2 + \alpha'\frac{V^2}{\kappa}L. \tag{2.7}$$

Both Vinen's original equation (2.1) and the alternative form (2.7) describe the dynamics of the second turbulent regime satisfactorily, but not the other ones. In fact, the stationary solutions of equation (2.1) are

$$L = 0 \qquad \text{and} \qquad L = \left(\frac{\alpha V}{\kappa\beta}\right)^2, \tag{2.8}$$

while the ones of equation (2.7) are

$$L = 0 \qquad \text{and} \qquad L = \frac{\alpha' V^2}{\beta\kappa^2}. \tag{2.9}$$

The solution $L = 0$ corresponds to the laminar regime, in which vortices are absent, and the other solutions to the turbulent regime. However, a simple stability analysis (see section 3.1) shows that in both equations (2.1) and (2.7) only the second solution is stable, for all values of the counterflow velocity V. This is not in agreement with experiments, where three successive states, laminar, turbulent I and turbulent II, are observed and there is also present a metastable laminar region. This shortcoming can be understood by observing that Vinen considered homogeneous superfluid turbulence. As remarked in [17], homogeneous turbulence can exist only in the case when the characteristic interline spacing, $\delta \sim L^{-1/2}$, is much smaller than the characteristic size, d, of the container. Therefore Vinen's original equation cannot describe the laminar and the turbulent TI regime, where L is relatively small.

In his final version, Vinen [18] introduced corrections connected with the dimension d of the channel, able to describe some features of the laminar–turbulent transition. This equation, however, is not able to describe the TI–TII transition. In [9], a different modification of Vinen's equation has been proposed, including corrections, both in the production and in the destruction term, depending on the ratio $L^{-1/2}/d$, which allows us to describe, in relatively good agreement with experimental results, the TI–TII transition, but it is not able to describe the laminar–turbulent transition.

3. Generalized vortex line density evolution equation

Our aim in this paper is to make a phenomenological extension of Vinen's equation (2.1) in order to write a single evolution equation able to describe the three stationary regimes observed in counterflow, in tubes of diameter d: a laminar regime at low V, the transition at the counterflow velocity V_{c1} from the laminar to the turbulent TI regime, the metastability region $V_{c1} < V < V'_{c1}$, the two turbulent regimes TI and TII at increasing values of V. In particular, we will consider two modifications: (*a*) we will include nonlinear production terms in the counterflow velocity V and (*b*) we will include corrections depending on $L^{-1/2}/d$, including the effects of the size of the capillary.

The nonlinear terms depending on V are introduced in order to describe the transitions. The motivation to include the diameter d of the capillary are twofold: phenomenological and theoretical. First, observe that the experimental results (critical velocities V_{c1} and V_{c2} and stationary states of turbulence) depend on the tube dimension. In particular, the dimensionless quantity $L^{-1/2}/d$ seems fundamental in the studies of superfluid turbulence. Indeed this term expresses the ratio of the average separation between vortices, $\delta \sim L^{-1/2}$, and the diameter of the channel. This term is negligible for fully developed turbulence, when L is high, but it is important when L is small; thus it is logical to expect that it will play a role in the transition from laminar to turbulent regimes.

The correction in δ/d may have some analogy with the presence of nonlocal terms in higher-order hydrodynamics, where the ratio l/d is considered, l being the mean free path. In this way, by using the Chapman–Enskog formalism or Grad formalism one obtains higher-order terms in l/d leading to higher-order hydrodynamics [15, 19]. Such terms are also of much current interest in molecular hydrodynamics [20] aiming to generalize hydrodynamics to the range of perturbations of wavelength comparable to the average interparticle separation. Since the mean-free path of the vortices in the tangle is of the order of their average separation, the mentioned analogy seems reasonable.

From a microscopic perspective, a possible motivation of the presence of d in the evolution equation of superfluid turbulence can be found in considerations relative to the phenomenon of vortex pinning and depinning [21, 1–4]. Vortices move and experience a reaction from the normal component; this couples the superfluid and the normal components and produces a 'mutual friction' between them. When the turbulence is developed, a tangle of quantized vortices is present in the channel. An important feature of vortex dynamics is the possibility of vortex reconnections, which change the topology of vortex lines: when two vortex filaments approach each other closely they reconnect [1–4]. As has been discussed in [21], in superfluid ^{4}He the vorticity is pinning. Awschalom and Schwarz [22] have shown that pinned vortices, presumably formed during the transition through the λ-point, are always found in the fluid, while freely moving vortices do not live a long time: either they get trapped on suitable prominences of the wall of the container or they lose their energy by interacting with the elementary excitations [23–26]. Schwarz's extensive numerical simulations [27–29] of vortex motion is based on the idea that a pinned vortex, perched on a hemispheric protuberance of radius b standing of a plane wall, unpins when the applied flow bends it very strongly. It then undergoes a reconnection and escapes from the wall to free space. From all these considerations it follows that the phenomenon of vortex pinning and unpinning is very important in the study of superfluid turbulence evolution, especially in the description of the laminar and turbulent TI regimes and in the transition from TI to TII turbulent regimes. We will suppose that all the phenomena described above are responsible, on the average, for corrections depending on $\frac{\delta}{d} \sim \frac{1}{L^{1/2}d}$ in Vinen's production and destruction terms.

The proposed equation for the evolution of vortex line density L, which we shall discuss in this section, including corrections depending on $\frac{\delta}{d} \sim \frac{L^{-1/2}}{d}$ and nonlinear corrections in V, is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \alpha(V,d)VL^{3/2}\left[1-\omega\frac{L^{-1/2}}{d}\right]+\alpha'\frac{V^2}{\kappa}L-\beta\kappa L^2\left[1+\omega'\frac{L^{-1/2}}{d}-\omega''\left(\frac{L^{-1/2}}{d}\right)^2\right], \tag{3.1}$$

where $\alpha(V,d)$, is approximately constant in the two turbulent regimes (α_{I} in the turbulence TI and α_{II} in TII), while it undergoes a steep change at the second critical velocity. Here and in the following, for the coefficient β, linked to the vortex cascade in the reconnection processes, we will choose expression $(2.6)_2$ obtained by Vinen.

We have thus new terms not present in Vinen's equation. In sections 3.1 and 3.2, we outline a possible physical motivation for the modifications made in the production term, while the modification of the destruction term will be discussed in more detail in section 3.3. In the successive sections we will show how equation (3.1) is useful to describe the known phenomenology of the three mentioned hydrodynamic regimes.

3.1. The laminar and the TI regimes

We start to study the transition from laminar to turbulent regime. This transition is a first-order one. In it L exhibits a discontinuity of the order of $L^{1/2}d \sim 2.5$, at a critical velocity V_{c1} which depends on the diameter of the capillary as $V_{c1} = c_1\kappa/d$; furthermore, the laminar state with $L = 0$ is found to be metastable from V_{c1} to another velocity V'_{c1}, not completely studied.

To describe this transition in a qualitative way, we include in Vinen's equation (2.1) corrections of order $\frac{\delta}{d} \sim \frac{L^{-1/2}}{d}$ and of order V^2, in the term responsible to the growth of vortices, obtaining

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \alpha_{\mathrm{I}} V L^{3/2}\left[1 - \omega\frac{L^{-1/2}}{d}\right] + \alpha'\frac{V^2}{\kappa}L - \beta\kappa L^2. \tag{3.2}$$

α_{I} being the value of $\alpha(V, d)$ in the laminar and TI regime (observe that the term with $\alpha_{\mathrm{I}}\omega V L/d$ had been considered by Vinen himself in early attempts [18, 30, 31]).

The steady-state solutions of (3.2) are $L_1 = 0$ and

$$L_{2,3} = \frac{1}{2}\frac{\alpha_{\mathrm{I}} V}{\beta\kappa}\left[1 \pm \sqrt{1 + \frac{4\beta\kappa}{\alpha_{\mathrm{I}} V}\left(\frac{\alpha'}{\alpha_{\mathrm{I}}}\frac{V}{\kappa} - \frac{\omega}{d}\right)}\right]. \tag{3.3}$$

The solutions $L_{2,3}$ (corresponding to + and − sign in (3.3), respectively) are real only for values of V higher than

$$V_{c1} = \frac{4\beta\alpha_{\mathrm{I}}\omega}{\alpha_{\mathrm{I}}^2 + 4\beta\alpha'}\frac{\kappa}{d}. \tag{3.4}$$

At the transition (when $V = V_{c1}$) there is a discontinuity in the value of L, which goes from $L = 0$ to

$$L_{c1}^{1/2} = \frac{2\alpha_{\mathrm{I}}^2\omega}{\alpha_{\mathrm{I}}^2 + 4\beta\alpha'}\frac{1}{d}. \tag{3.5}$$

The stability of these solutions may be studied from the equation for a perturbation δL of L, given by

$$\frac{\mathrm{d}\delta L}{\mathrm{d}t} = -\left[2\beta\kappa L - \frac{3}{2}\alpha_{\mathrm{I}} V L^{1/2} - \alpha'\frac{V^2}{\kappa} + \alpha_{\mathrm{I}}\omega\frac{V}{d}\right]\delta L, \tag{3.6}$$

from which it follows that the solution $L_1 = 0$ is stable up to a value of V'_{c1} given by

$$V'_{c1} = \frac{\alpha_{\mathrm{I}}\omega\kappa}{\alpha' d}. \tag{3.7}$$

The solution L_2 is stable where it exists, while L_3 is unstable. Further it is $V_{c1} < V'_{c1}$, as one verifies immediately.

These results are in qualitative agreement with experiments: in fact one recovers the dependence $V_{c1} \sim \kappa/d$ and $L_{c1} \sim 1/d$, the discontinuity in the transition from laminar to turbulent low-density TI regime and the existence of a metastability region of the laminar state.

3.2. The TI–TII transition

It has been seen that the evolution equation (3.2) describes the transition from laminar to turbulent TI regime, namely, the critical velocity V_{c1}, the discontinuity of L at this value, and the range of metastability of the laminar solution. However, it does not describe the transition TI–TII, taking place at the second critical velocity V_{c2}.

In [9], a modification of Vinen's equation has been proposed, including corrective terms, depending on the ratio $L^{-1/2}/d$. This equation allows us to describe the TI–TII transition, although it leads to an underestimation of the critical velocity V_{c2}. This could indicate that the nonlocal terms may have a form different from the one assumed in that work.

Here, to describe the TI–TII transition, we follow a different route: we introduce heuristically a dependence on Vd/κ of the coefficient α in the first term of (3.1), responsible for the growth of L. Indeed, as we have already observed, experimentally one sees that at this transition the coefficient α, which characterize the slope of the stationary solution, undergoes a step from the value α_{I} in the TI regime to a value α_{II} in the TII regime. Therefore, the coefficient α must depend on V, L and d. In [9] we have supposed α dependent on L and on d; here we will suppose α dependent on V and on d, making the following assumption:

$$\alpha(V,d) = \alpha_{c2}\left(1 + c\tanh\left[A\left(\frac{Vd}{\kappa} - C\right)\right]\right), \tag{3.8}$$

c, C and A being dimensionless constants in such a way that, for $V \ll V_{c2}$, $\alpha \simeq \alpha_{\mathrm{I}} = \alpha_{c2}(1-c)$ and for $V \gg V_{c2}$, $\alpha \simeq \alpha_{\mathrm{II}} = \alpha_{c2}(1+c)$, α_{c2} being the value of $\alpha(V,d)$ at the second transition. The constant C is related to V_{c2} by

$$V_{c2} = \frac{C\kappa}{d}, \tag{3.9}$$

which gives the way to measure the coefficient C, whereas $2c = \alpha_{\mathrm{II}} - \alpha_{\mathrm{I}}$ gives the size of the step of α near V_{c2}. The constant A characterizes the size of the domain of V in which the transition from α_{I} to α_{II} is produced.

Now, we discuss some details of the TI–TII transition and the physical motivation for (3.8). A possible explanation of the TI–TII transition has been proposed by Melotte and Barenghi [32]. These authors have shown that an initially laminar normal fluid may in some circumstances become unstable in the presence of turbulent superfluid, so that both fluids may become turbulent. This could have an effect on the observed mutual friction and it might give rise to the observed different regimes. Therefore they infer that in the TI regime only the superfluid component is turbulent, while in the TII regime both the superfluid and the normal components become turbulent. However, their linear stability analysis does not capture the correct temperature dependence [32, 18]. Furthermore, Schwarz in his numerical simulations [27–29], which are obtained under the assumption that the flow of the normal fluid is laminar and spatially uniform, provided an excellent confirmation of Vinen's equation in the TII state; so that the suggestive hypothesis by Melotte and Barenghi appears not completely confirmed by the numerical simulations of Schwarz and experiments.

A possible different description of the phenomenon, which of course must have adequate theoretical and experimental confirmations, which we have sketched in figure 1, can be the following: in the laminar regime, when $V < V_{c1}$, in the channel there are present vortex lines (the so-called 'remnant vortices' formed when helium was cooled through the λ-point or during previous turbulent flows [22], figure 1(a)), which are strongly pinned to protuberances of the walls and slightly bent by the heat flux; as the counterflow velocity grows, these lines are bent and their length increases, even if most of them remain pinned to the channel. Helical waves may be propagated in these (initially rectilinear) vortices which remain strongly pinned

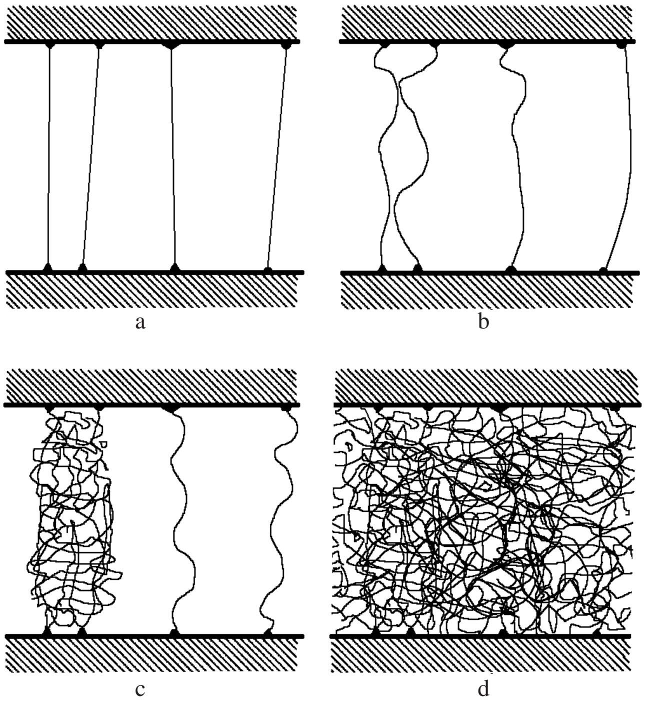

Figure 1. A sketch of how the vortices behave, following the discussion in section 3.2: (a) $V = 0$; (b) laminar regime ($V < V'_{c1}$); (c) turbulent TI regime ($V_{c1} < V < V_{c2}$); (d) turbulent TII regime ($V > V_{c2}$).

on prominences of the wall of the channel (figure 1(b)). When the counterflow velocity grows, the separation between successive coils of these helices reduces. When the counterflow velocity reaches the first critical velocity V_{c1}, in correspondence of these waves, small localized arrays of quantized vortices appear, which result in being polarized in the (mean) direction of the equilibrium configuration of the initial vortex (figure 1(c)). Sometimes, depending on the initial configuration of the vortices existing in the channel in the absence of heat flux, these localized arrays may be generated for values of V beyond V_{c1}, so that a region of metastability of the laminar regime arises. In this picture, the TI turbulent regime is an inhomogeneous and locally polarized state (roughly isotropic in the whole, or slightly non-isotropic in the direction of the heat flux). When the counterflow reaches the critical value V_{c2} this state becomes unstable and the flow undergoes a transition to the fully developed turbulent regime TII (figure 1(d)). The critical velocity V_{c2} indicates the definitive breakdown of these localized polarizations and the transition to the homogeneous slightly non-isotropic state TII. This is in accord with the experimental observations [33] and computer simulations [28, 29] which show that in counterflow turbulence the tangle is roughly isotropic or slightly non-isotropic, and this anisotropy is independent on the counterflow velocity; it is also in accord with recent numerical investigations, where a new form of energy cascade of helical waves on vortex filaments is described [34], and experimental results [35] which suggest proposing Kelvin waves as a depinning mechanism.

The transition from TI to TII, which we have modelled with the function (3.8), may be due to the phenomenon just described. As we will see in section 4, in the laminar and in the turbulent TI regime the interaction with the boundary is prevalent, and therefore the terms dependent on d, while in the TII regime the terms dependent on the heat flux, linked to the counterflow velocity V, are prevalent. In the transition region, which has been described with the function (3.8) there is a competition between order and disorder: $1/d$ contributes to the order, i.e. to the decrease of vortex lines, while V contributes to the disorder, i.e. to their increase.

3.3. Corrections to the destruction term

The intuitive interpretation of Vinen's corrective term in equation (3.2) is that the vortex generation mechanism is inactive within a characteristic distance $L^{-1/2}$ from the wall. This explains the choice of a negative sign in the new term $\omega L^{-1/2}/d$ in the first term (production term) on the right-hand side of (3.1); but this idea left unchanged the decay term in Vinen's second proposal [18]. However, the walls play an important role in vortex dynamics, and also in the decay processes. Indeed, Schwarz, through extensive numerical simulations [28], has studied the interaction between a single vortex line and a boundary. He has shown that this interaction occurs when the vortex lines approach the wall closer than a characteristic distance Δ_c, defined as the critical distance between the loop and its image. If the distance Δ between a vortex ring (or more in general, a vortex loop) and the surface is smaller than Δ_c, the boundary traps the vortex, which is reconnected to the surface. Further, the nearest part of a vortex loop travelling parallel to a plane boundary is retarded by the boundary field. In contrast, if the distance Δ between the surface and the vortex loop is greater than Δ_c, the vortex motion is only weakly affected by the boundary. Obviously, if the dimension d of the channel is small, the influence of the walls on the decay processes cannot be neglected and we should take account of the repeated images too. We propose therefore to modify the term responsible for the vortex decay introducing in it a corrective term, depending on $\langle\Delta\rangle/d$, and we substitute (2.6) with

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_d = -\beta\kappa L^2\psi\left(\frac{\langle\Delta\rangle}{d}\right), \tag{3.10}$$

where brackets denote average; also, in this case, we can suppose that $\langle\Delta\rangle$ is proportional to $L^{-1/2}$: $\langle\Delta\rangle = \lambda L^{-1/2}$; therefore, we shall modify Vinen's destruction term (2.6), introducing in it a quadratic dependence on $L^{-1/2}/d$:

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_d = -\beta\kappa L^2\left[1+\omega'\frac{L^{-1/2}}{d}-\omega''\left(\frac{L^{-1/2}}{d}\right)^2\right]. \tag{3.11}$$

With this choice of the signs in the coefficients, the destruction term is higher than the usual Vinen proposal when the typical vortex–wall separation Δ is less than $\Delta_c = \lambda(\omega'/\omega'')d$, whereas for higher values it is lower. One can write (3.11) as

$$\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right]_d = -\beta\kappa L^2\left[1-h\frac{L^{-1/2}}{d}\frac{\langle\Delta-\Delta_c\rangle}{d}\right] \tag{3.12}$$

where $h = \omega''/\lambda$. This provides a phenomenological description of the qualitative ideas mentioned. Of course, equation (3.11) (or (3.12)) is only a first approximation of the unknown function (3.10), valid if $L^{-1/2}/d$ is not too high. Nevertheless, as we will show in section 6, the introduction of these corrective terms, together with the corrections in the production terms, allows us to describe the stationary solutions in counterflow situations and produces a decay of superfluid turbulence, in the absence of counterflow velocity, slower than that obtained using the original Vinen equation.

From a physical point of view, the maximum value of the ratio Δ/d (vortex–wall distance/diameter of the tube) is $1/2$, and the proposed form for the coefficient of the destruction term is only indicative. However, the idea of a change of the sign of such term, becoming then a production term not related to the velocity, could be worthy of exploration in connection with the recent experimental discovery of a velocity-independent transition to superfluid turbulence [36]. This is certainly an interesting topic, but it is beyond the scope of the present paper.

4. Stationary solutions of the generalized Vinen equations

In this section we will study the stationary solutions of the full equation (3.1), which we rewrite as

$$\frac{dL}{dt} = -\beta\kappa L^2 + \left[\alpha\left(\frac{Vd}{\kappa}\right)V - \omega'\beta\frac{\kappa}{d}\right]L^{3/2} + \left[\alpha'\frac{V^2}{\kappa} - \omega\alpha\left(\frac{Vd}{\kappa}\right)\frac{V}{d} + \omega''\beta\frac{\kappa}{d^2}\right]L, \quad (4.1)$$

with α expressed by (3.8), and their corresponding stability. We will show that the stationary solutions of (4.1) are able to describe, in accord with experimental results, all the stationary regimes present in counterflow superfluid turbulence.

To this purpose, we will perform a change of variables obtaining a dimensionless equation for the evolution of the vortex line density. Putting

$$L^{1/2}d = y, \qquad Vd\kappa^{-1} = x, \quad (4.2)$$

we obtain

$$\frac{dy}{dt} = \frac{\beta\kappa}{2d^2}\left[-y^3 + \left(H(x)x - \omega'\right)y^2 + \left(H'x^2 - \omega H(x)x + \omega''\right)y\right], \quad (4.3)$$

with

$$H(x) = \frac{\alpha(x)}{\beta}, \qquad H' = \frac{\alpha'}{\beta}, \quad (4.4)$$

and $C = V_{c2}d/k = x_{c2}$.

The nonzero stationary solutions of equation (4.3) are the solutions of the equation

$$-y^2 + \left(H(x)x - \omega'\right)y + H'x^2 - \omega H(x)x + \omega'' = 0. \quad (4.5)$$

The function $H(x)$ will assume (approximately) the value $H_I = \alpha_I/\beta$ in the TI regime and (approximately) the value $H_{II} = \alpha_{II}/\beta$ in the TII regime. As a consequence, equation (4.5) can be approximated with the hyperbola

$$-y^2 + \left(H_I x - \omega'\right)y + H'x^2 - \omega H_I x + \omega'' = 0, \quad (4.6)$$

in the TI region, and with the hyperbola

$$-y^2 + \left(H_{II} x - \omega'\right)y + H'x^2 - \omega H_{II} x + \omega'' = 0, \quad (4.7)$$

in the TII region.

The stationary solutions of equation (4.3) are the solution $y_1 = 0$, corresponding to the laminar regime ($L_1 = 0$), and the two functions

$$y = y_2(x) = \tfrac{1}{2}\left[H(x)x - \omega' + \sqrt{(H(x)x - \omega')^2 + 4\left(H'x^2 - \omega H(x)x + \omega''\right)}\right] \quad (4.8)$$

$$y = y_3(x) = \tfrac{1}{2}\left[H(x)x - \omega' - \sqrt{(H(x)x - \omega')^2 + 4\left(H'x^2 - \omega H(x)x + \omega''\right)}\right] \quad (4.9)$$

corresponding respectively to the full turbulent regime (TI and TII) and to an unstable solution, not observed experimentally.

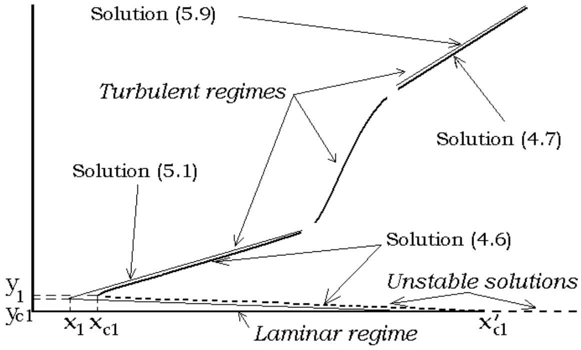

Figure 2. Qualitative description of the stationary solutions of equation (4.3), and of the approximations made substituting solution (4.8) with the approximate solution (5.1), (5.4) and (5.9).

5. Some simplifying hypotheses

In order to determine numerical values for the coefficients appearing in equation (4.3), in this section we will make some approximations. In figure 2 a qualitative description of the stationary solutions of equation (4.3), and of the approximations made in this section are shown.

5.1. The turbulent TI regime

First, we study the TI regime. Under the hypothesis that the values considered are far from the TI–TII transition region, we can approximate equation (4.5) with equation (4.6).

Because the experimental data show that the stationary solution in the TI regime can be approximated with a straight line, we can see that equation (4.8) is very near to the asymptote of the equation

$$y = \frac{1}{2}\left[\left(H_{\rm I} + \sqrt{H_{\rm I}^2 + 4H'}\right)x - \left(\frac{H_{\rm I}(2\omega + \omega')}{\sqrt{H_{\rm I}^2 + 4H'}} + \omega'\right)\right] = a_1 x - b_1, \quad (5.1)$$

so we obtain

$$H_1 + \sqrt{H_{\rm I}^2 + 4H'} = 2a_1, \qquad H_{\rm I}\omega + \omega' a_1 = b_1\sqrt{H_{\rm I}^2 + 4H'}. \quad (5.2)$$

In the variables V and L, equation (5.1) can be written

$$L_{\rm I}^{1/2} = \gamma_1 V - \frac{1}{d} b_1, \quad (5.3)$$

where $\gamma_1 = a_1/\kappa$. This relation has been obtained experimentally in several works, so the values of γ_1 and b_1 are known from experiments. To give an idea of the order of magnitude of the mentioned physical quantities, table 1 gives the values of a_1 and b_1 for $T = 1.5$ and 1.7 K obtained by using the experimental data of Martin and Tough [5].

In the same way, we can say that for these values of V the function (4.9) relative to the unstable solution, will be approximated by the straight line

$$y = \frac{1}{2}\left[-\left(\sqrt{H_{\rm I}^2 + 4H'} - H_{\rm I}\right)x + \left(\frac{H_{\rm I}(2\omega + \omega')}{\sqrt{H_{\rm I}^2 + 4H'}} - \omega'\right)\right] = -mx + n, \quad (5.4)$$

Table 1. Values of a_1, b_1, a_2, b_2 and of x_{c1} and x_{c2}, obtained from the data reported in [5].

T (K)	a_1	b_1	a_2	b_2	x_{c1}	x_{c2}
1.5	0.076	7.25	0.139	9.99	127.07	219
1.7	0.091	6.92	0.176	8.18	97.74	186

so we obtain

$$\sqrt{H_{\rm I}^2+4H'}-H_{\rm I}=2m, \qquad H_{\rm I}\omega+\omega' m=n\sqrt{H_{\rm I}^2+4H'}. \tag{5.5}$$

From equations (5.1) and (5.5) one obtains, finally,

$$H_{\rm I}=a_1-m, \qquad H'=a_1 m, \qquad \omega'=b_1-n, \qquad \omega=\frac{a_1 n+b_1 m}{a_1-m}. \tag{5.6}$$

The two straight lines (5.1) and (5.4) meet in the centre (x_1, y_1) of the hyperbola (4.6); this point is very near to the point (x_{c1}, y_{c1}) corresponding to the values of V and L characterizing the transition to turbulent regime. If, in a first approximation, we suppose that in this region the hyperbola degenerates into the two straight lines of equations (5.1) and (5.5), one has

$$x_{c1}=x_1=\frac{b_1+n}{a_1+m}, \qquad y_{c1}=y_1=\frac{a_1 n-b_1 m}{a_1+m}, \tag{5.7}$$

the straight line (5.4) contains the end-point $(x'_{c1}, 0)$ of the metastability region of the laminar regime, the coefficient ω'' assumes the value $\omega''=b_1 n$ and it results that

$$m=\frac{y_1}{x'_{c1}-x_1}, \qquad n=\frac{x'_{c1}y_1}{x'_{c1}-x_1}>0. \tag{5.8}$$

In this first rough comparison with experimental results we will make this hypothesis.

5.2. *The turbulent TII regime*

The next question is how turbulence II behaves for high values of L, i.e. in the asymptotic fully developed TII regime. If the values considered are far from the transition region, we can approximate equation (4.5) with equation (4.7). We determine, therefore, the equation of the asymptote with positive slope of hyperbola (4.7). We get

$$y=\frac{1}{2}\left[\left(H_{\rm II}+\sqrt{H_{\rm II}^2+4H'}\right)x-\left(\frac{H_{\rm II}(2\omega+\omega')}{\sqrt{H_{\rm II}^2+4H'}}+\omega'\right)\right]=a_2x-b_2, \tag{5.9}$$

so we obtain

$$H_{\rm II}+\sqrt{H_{\rm II}^2+4H'}=2a_2, \qquad H_{\rm II}\omega+\omega' a_2=b_2\sqrt{H_{\rm II}^2+4H'}. \tag{5.10}$$

In the variables V and L, equation (5.10) is written

$$L_{\rm II}^{1/2}=\gamma_2 V-\frac{1}{d}b_2, \tag{5.11}$$

where $\gamma_2=a_2/\kappa$. Again, this form of $L^{\frac{1}{2}}$ is well documented in the literature as giving a good description of experimental results. Table 1 gives the values of a_2 and b_2 for $T=1.5$ and 1.7 K as obtained by data of Martin and Tough [5].

From $(5.10)_1$, using equations (5.6), we obtain the value of $H(x)$ in the asymptotic turbulent TII regime:

Table 2. Values of $H_{\rm I}$, $H_{\rm II}$, H', ω, ω' and ω'' from this work, obtained from the data reported in [5].

T (K)	$H_{\rm I}$	$H_{\rm II}$	H'	ω	ω'	ω''
1.5	0.0664	0.134	0.000 725	5.178	3.634	25.1
1.7	0.0830	0.172	0.000 731	3.560	4.259	17.3

$$H_{\rm II} = a_2 - \frac{a_1}{a_2} m, \tag{5.12}$$

while $(5.10)_2$ furnish a link between the parameter b_2 and the other coefficients. Using (5.6) and (5.10), one obtains the following equation in the unknowns m and n:

$$(a_2^2 - a_1 m)(a_1 n + b_1 m) - (a_2^2 + a_1 m)(a_1 - m) b_2 + a_2^2 (b_1 - n)(a_1 - m) = 0. \tag{5.13}$$

Substituting in this latter equation expressions (5.8) of m and n, we obtain finally the end-point $x'_{\rm c1}$ of the metastability region, as a function of the quantities a_1, b_1, a_2, b_2, which are known from experimental data and of the coordinates of the centre (x_1, y_1) of the hyperbola (4.6):

$$x'_{\rm c1} = x_1 + y_1 K \tag{5.14}$$

where K is the positive solution of

$$a_1 a_2^2 K^2 - y_1 (y_1 + b_1)(a_2^2 - a_1^2) K - (a_2^2 - a_1^2) y_1^2 x_1 - a_1 (b_2 - b_1) y_1^2 = 0. \tag{5.15}$$

Choosing, in a first approximation, $x_1 = x_{\rm c1}$ (and $y_1 = a_1 x_1 - b_1$), though being aware that the result obtained will not be accurate, using the experimental data by Martin and Tough [5], we have calculated the value of $x'_{\rm c1}$, furnished by equation (5.14). Finally, using this result, we have determined the values of m and n and of the coefficients $H_{\rm I}$, $H_{\rm II}$, H', ω and ω'. These values are reported in table 2.

In figure 3 are reported the experimental data of [5] and our theoretical predictions; for the coefficient A we have taken $A = 0.05$ at $T = 1.5$ K and $A = 0.25$ at $T = 1.7$ K.

Taking in mind the simplifying hypotheses made in this section, the good agreement between our macroscopic description and experimental observations suggest that the proposed phenomenological model is a good approximation of a theoretical unknown model, which, in the approximations made, ought to reduce itself to equation (4.1).

5.3. Approximate equations in the turbulent TI and TII regimes

If we neglect the metastability of the laminar regime and the small step of L at the critical velocity $V_{\rm c1}$, we can approximate equation (4.1) with two simpler equations, which separately describe the two regimes TI and TII.

Recalling that in the TI region the hyperbola (4.6) is very near to the asymptotes (5.1) and (5.4), we can approximate equation (4.3) in the following way:

$$\frac{{\rm d}y}{{\rm d}t} \simeq \frac{\beta\kappa}{2d^2} [-y(y - a_1 x + b_1)(y + mx - n)]. \tag{5.16}$$

We can substitute the function $(y + mx - n)$, which varies very little in the TI region, with a constant c (for example the value $c = y_{\rm c1} = L_{\rm c1}^{1/2} d$). We obtain, in this way,

$$\frac{{\rm d}y}{{\rm d}t} \simeq y_{\rm c1} \frac{\beta\kappa}{2d^2} [-y(y - a_1 x + b_1)], \tag{5.17}$$

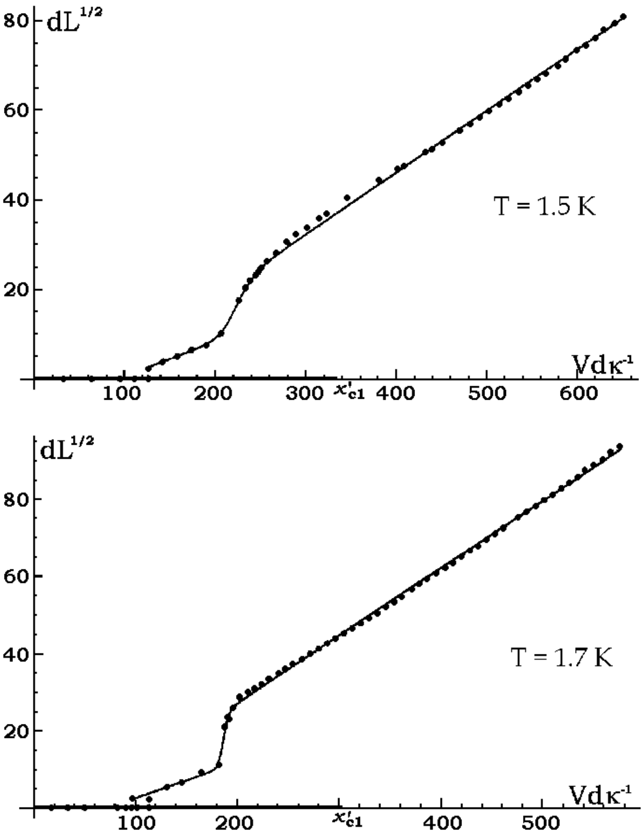

Figure 3. $y = L^{1/2}d$ as a function of $x = Vd\kappa^{-1}$ at $T = 1.5$ and 1.7 K. Data are from Martin and Tough [5]. Lines are determined from this work (equation (4.8) with $A = 0.05$ at $T = 1.5$ K and $A = 0.25$ at $T = 1.7$ K). The (meta)stability region of the laminar regime is also indicated.

with a_1 and b_1 expressed by equations (5.2). In the variables L and V, equation (5.17) can be written

$$\frac{\mathrm{d}L}{\mathrm{d}t} \simeq \beta\kappa L_{c1}^{1/2}\left[-L^{3/2} + \left(\gamma_1 V - \frac{b_1}{d}\right)L\right], \tag{5.18}$$

with $\gamma_1 = a_1/\kappa$. This equation differs very little from the equation

$$\frac{\mathrm{d}L}{\mathrm{d}t} \simeq \beta\kappa\left[-L^2 + \left(\gamma_1 V - \frac{b_1}{d}\right)L^{3/2}\right], \tag{5.19}$$

proposed by Vinen in [19], and has the same stationary solution (5.3).

We will finally determine a simplified equation for the evolution of L in the asymptotic fully developed TII regime. For high values of L and V, we can substitute $\alpha(x)$ with its value α_{II} in the TII regime and we can neglect in (4.1) the term depending on the dimension d of the tube, thus obtaining the following approximate equation:

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2 + \alpha_{\mathrm{II}} V L^{3/2} + \alpha'\frac{V^2}{\kappa}L. \tag{5.20}$$

The stationary solution of this equation is

$$L^{1/2} = \frac{\alpha_{\mathrm{II}} + \sqrt{\alpha_{\mathrm{II}}^2 + 4\beta\alpha'}}{2\beta}\frac{V}{\kappa} = \frac{H_{\mathrm{II}} + \sqrt{H_{\mathrm{II}}^2 + 4H'}}{2}\frac{V}{\kappa}. \tag{5.21}$$

Observe that (5.21) modifies the coefficient in the solution (2.8) of Vinen's original equation (2.1), introducing in it the coefficient α'. A glance at the values of H_{II} and H' reported in table 2 shows that, in the TII regime, the influence of the coefficient α', linked to the quadratic term in equation (4.1), is negligible, and we recover Vinen's original equation.

6. Vortex decay towards a quiescent state

As a new illustration of the possible physical interest of (3.1), we consider the decay of vorticity in counterflow superfluid turbulence, after the heat flux, proportional to V, is suddenly set to zero. According to Vinen's equation (2.1), such decay is described by

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2, \tag{6.1}$$

thus leading to

$$\frac{1}{L(t)} = \frac{1}{L_0} + \beta\kappa t. \tag{6.2}$$

This solution corresponds to the decay of a homogeneous vortex tangle, which occurs when L is high. However, comparison with experimental data [37] indicates that the decay of L is much slower than this prediction. We will study here how nonlocal terms in $L^{1/2}/d$, increasingly important as L is lowered, may contribute to the mentioned slowing down of the decay. With this aim, we now study decay processes using equation (3.1), which is a better model to describe nonhomogeneous vortex decay.

When $V = 0$, equation (3.1) leads to

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\beta\kappa L^2\left[1 + \omega'\frac{L^{-1/2}}{d} - \omega''\left(\frac{L^{-1/2}}{d}\right)^2\right]. \tag{6.3}$$

Putting

$$z = \frac{1}{L^{1/2}d}, \qquad \omega' = \gamma_2 - \gamma_1, \qquad \omega'' = \gamma_1\gamma_2 \qquad (\gamma_1 > 0, \gamma_2 > 0), \tag{6.4}$$

the solution of equation (6.3) is

$$\gamma_2 \ln|1 - \gamma_1 z| + \gamma_1 \ln|1 + \gamma_2 z| = -\gamma_1\gamma_2(\gamma_1 + \gamma_2)\frac{\beta\kappa}{2d^2}t + c, \tag{6.5}$$

where $\gamma_1 + \gamma_2 = \sqrt{\omega'^2 + 4\omega''}$. Denoting with L_0 the initial value of L, one obtains, therefore,

$$\ln\left|\frac{1 - \gamma_1\frac{1}{\mathrm{d}L^{1/2}}}{1 - \gamma_1\frac{1}{\mathrm{d}L_0^{1/2}}}\right|^{\gamma_2}\left|\frac{1 + \gamma_2\frac{1}{\mathrm{d}L^{1/2}}}{1 + \gamma_2\frac{1}{\mathrm{d}L_0^{1/2}}}\right|^{\gamma_1} = -\gamma_1\gamma_2(\gamma_1 + \gamma_2)\frac{\beta\kappa}{2d^2}t. \tag{6.6}$$

The left-hand side of (6.6), for small values of z (i.e. for high values of L), becomes $-\frac{1}{2}\gamma_1\gamma_2(\gamma_1 + \gamma_2)z^2$; one obtains, therefore, the solution

$$\frac{1}{L} \simeq \frac{1}{L_0} + \beta\kappa t. \tag{6.7}$$

As one sees, for small values of t, equation (6.7) coincides with (6.2).

For higher values of t, to second order in z, the solution of (6.3) is

$$z^2 \simeq \frac{1}{B}\left[1 - c_1 \exp\left(-\frac{B}{d^2}\beta\kappa t\right)\right], \tag{6.8}$$

Table 3. Values of β from [37] and of y_{c1} from [5]. Values of B, z_∞ and $y_\infty = L_\infty^{1/2}/d$ from this work.

T (K)	β	y_{c1}	B	z_∞	L_∞^{-1}	$y_\infty = dL_\infty^{1/2}$
1.5	0.78	∼2.5	5.36	0.285	0.000 81	3.51
1.7	1.30	∼2.5	4.70	0.393	0.001 54	2.52

where we have put $\gamma_1\gamma_2(\gamma_1+\gamma_2) = \omega''\sqrt{\omega'^2+4\omega''} = 2B$. The numerical values for B have been reported in table 3. One obtains for L,

$$\frac{1}{L} \simeq \left(\frac{1}{L_0} - \frac{d^2}{B}\right)\exp\left(-\frac{B}{d^2}\beta\kappa t\right) + \frac{d^2}{B}. \tag{6.9}$$

Finally, we observe that physical solutions $z = z(t)$ of equation (6.3) (implicitly defined by (6.6)) have the non-vanishing asymptotic value

$$z_\infty = \frac{\omega' + \sqrt{\omega'^2+4\omega''}}{2\omega''}. \tag{6.10}$$

The values of z_∞ and the corresponding values of L_∞, obtained using those reported in table 2 for ω' and ω'', are shown in table 3.

The fact that the asymptotic value of L is different from zero is a satisfactory feature in comparison with experimental data; in fact, it is known that, after the decay, a small fraction of vortices still survives, pinned to the walls.

Recall now that at the transition laminar → turbulence (in stationary counterflow) there is a discontinuity in the value of L, which goes from $L = 0$ to $L_{c1}^{1/2} = y_{c1}/d$. The asymptotic values of the quantity $y = L^{1/2}d$ and the experimental values of y_{c1} determined by Martin and Tough [5] are reported in table 3. As one sees, there is a sufficient agreement between values of y_∞ and values of y_{c1}, especially at $T = 1.7$ K.

The plots of the solution (6.6) of (6.3), choosing as initial point the value $\frac{1}{L_0} = 1.52\times10^{-6}\,\text{cm}^2$ at $T = 1.5$ K (corresponding to $y_0 = 81$) and the value $\frac{1}{L_0} = 1.13\times10^{-6}\,\text{cm}^2$ at $T = 1.7$ K (corresponding to $y_0 = 94$) at $T = 1.7$ K, are reported in figure 4.

As has been shown in equation (6.10) and in figure 4, a small number of vortices are always present in the channel. The possibility that pinned vortices, formed during the transition through the λ-point, are always found in the fluid, also in the absence of turbulence, has been advanced in the literature [22, 1] and will be explored in a further work, as they may play a relevant role as initiators of turbulence.

According to Schwarz and Rozen [37], the slow decay may be related to viscous effects in the normal fluid which, during its deceleration, is not immediately followed by the superfluid. This produces a small difference between $\mathbf{v}_n$ and $\mathbf{v}_s$ capable of sustaining the vortex tangle for a long time. Recall that the characteristic time of decay of a viscous fluid in a channel of diameter d is of the order of $\tau = \frac{d^2}{\nu}$, ν being its kinematic viscosity. In equation (3.1), an analogous slow relaxation could be associated to the last term, with a characteristic time $\tau' = \frac{d^2}{\kappa\beta\omega''}$. Thus here, $\kappa\beta\omega''$ plays a role similar to the kinematic viscosity of a normal fluid and it could yield a behaviour for the decay analogous to the one suggested by Schwarz and Rozen [37], following the same scaling with d as viscous effects.

7. Conclusions

In summary, we have indicated in this paper that a direct extension of Vinen's original equation greatly enlarges the ability to describe the phenomena found in superfluid counterflow experiments. The suggested generalization (3.1) introduces nonlinear terms in the counterflow

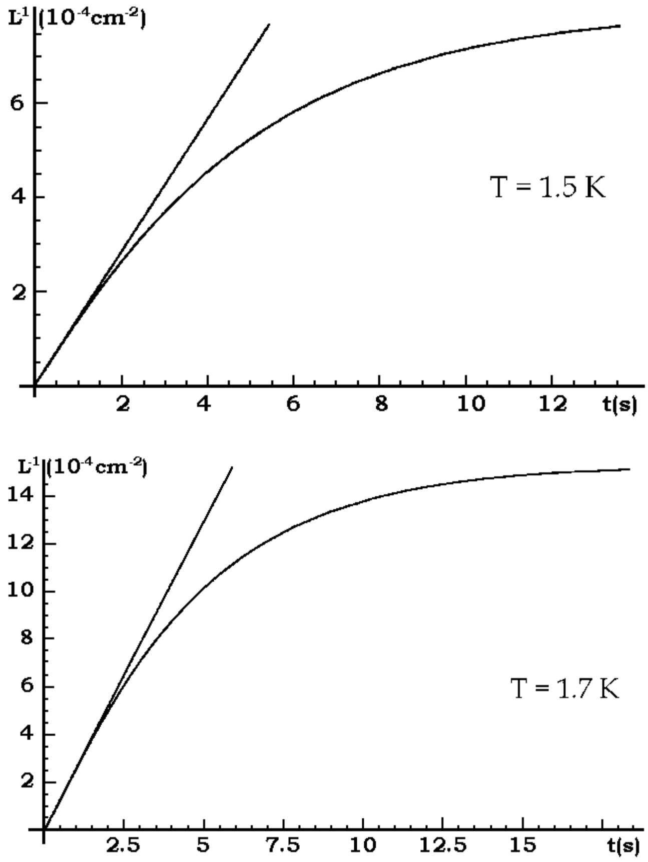

Figure 4. Plots of the solution (6.6) of equation (6.3) ($1/L$ as function of t), at $T = 1.5$ K and $T = 1.7$ K, using the values of ω', ω'' and β reported in tables 2 and 3. The initial values are $\frac{1}{L_0} = 1.52 \times 10^{-6}$ cm^2 at $T = 1.5$ K and $\frac{1}{L_0} = 1.13 \times 10^{-6}$ cm^2 at $T = 1.7$ K. The straight line is solution (6.2).

velocity V and incorporates corrections depending on δ/d, δ being the average intervortex separation and d the diameter of the channel. This allows us to describe the laminar regime ($L = 0$) including the metastability region, the transition from laminar regime to turbulent TI regime (characterized by the critical value V_{c1} of the velocity, and the value L_{c1} of the discontinuity in the line density) and the dependence of L with V and d for high values of L, namely, for well-developed turbulence TII regime. The transition from TI to TII regimes is phenomenologically described introducing a steep variation in a coefficient. Furthermore, the inclusion of corrective terms depending on δ/d in the destruction term of Vinen's equation yields a slower decay of the counterflow turbulence than the classical local description, where such a ratio is negligible.

The inclusion of d into the dynamic equations for turbulence cannot be justified only on dimensional arguments or on analogies with higher-order hydrodynamics, but on the physical basis provided by the pinning and the unpinning of vortex lines on the surface on the tube. So far a sufficiently detailed quantitative information of these phenomena is not available, and it is thus not possible to justify on a microscopic basis the new terms in (3.1). For the moment, their reliability must be tested by comparison of their predictions with experimental results.

However, we feel that the generalized equation (3.1) is a useful and natural extension of Vinen's original equation, which deserves some interest in view of its wider range of applications, and which enhances the relevance of Vinen's equation which, in this most general setting, shows more clearly its limits of validity. The final test for such a generalized equation will be its practical usefulness to describe experiments.

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